COMBINATORIAL ANALYSIS

PROBLEM SET 2 SOLUTIONS (MIT, FALL 2021)

Problem 1. *Find an explicit simple formula for the number of compositions of $2n$ whose largest part is n.*

Proof. First, note that there is exactly 1 composition of $2n$ into exactly 2 parts whose largest part is n, namely (n, n). We now count the number of compositions of size greater than 2.

It is known from lecture that the number of k-compositions of n is $\binom{n-1}{k-1}$. Further, if $k > 2$, then at most 1 element can equal n and the remaining elements sum to n. Therefore, a valid k-composition of $2n$ whose largest part is n has k spots to place n and $\binom{n-1}{k-2}$ ways to choose the other elements. Summing this over all values of k from 3 to $n+1$ gives:

$$\sum_{k=3}^{n+1} k\binom{n-1}{k-2} = \sum_{k=1}^{n-1}\left(k\binom{n-1}{k} + 2\binom{n-1}{k}\right) = (n-1)2^{n-2} + 2(2^{n-1}-1) = (n+3)2^{n-2} - 2.$$

Note that the identity $\sum_{k=1}^{n-1} k\binom{n-1}{k} = (n-1)2^{n-2}$ comes from the argument that both sides count the number of ways to pick a president on a committee of arbitrary size from $n-1$ people. Adding in the 1 composition of size 2 gives an answer of $(n+3)2^{n-2} - 1$. □

Problem 2. *Let $F(n)$ be the number of partitions of $[n]$ that do not contain any block of size 1. Prove combinatorially that $B(n) = F(n) + F(n+1)$, where $B(n)$ is the n-th Bell number.*

Proof. First, it is known that $B(n)$ counts the number of ways to partition a set of exactly n elements. Therefore, $B(n) - F(n)$ counts the number of partitions of $[n]$ that contain at least 1 block of size 1. We now seek to form a bijection between the number of partitions of $[n+1]$ containing no blocks of size 1 and the partitions of $[n]$ that contain at least 1 block of size 1. This will evidently show that $F(n+1) = B(n) - F(n)$.

For each $n \in \mathbb{N}$, let X_n be the set of partitions of $[n]$ that have no blocks of size 1 and Y_n be the set of partitions of $[n]$ that have at least 1 block of size 1. Define a function $f : X_{n+1} \to Y_n$ as follows. Consider an element $x \in X_{n+1}$, and suppose x represents the partition $a_1 \cup a_2 \cup ... \cup a_k$. It is know that $|a_i| > 1 \forall i$. Without loss of generality, the element $n+1$ is in set a_1. Suppose that $a_1 = \{r_1, r_2, ..., r_\ell, n+1\}$, and consider the partition $y = \{r_1\} \cup \{r_2\} \cup ... \cup \{r_\ell\} \cup a_2 \cup ... \cup a_k$. This is a valid partition of $[n]$ with at least 1 block of size 1 since $\ell \geq 1$. Therefore $y \in Y_n$ and is unique to each x.

For the reverse direction, take $f^{-1} : Y_n \to X_{n+1}$. Let $y \in Y_n$ so that the elements in a block by themselves are $r_1, r_2, ..., r_\ell$. If $y = \{r_1\} \cup \{r_2\} \cup ... \cup \{r_\ell\} \cup a_1 \cup a_2 \cup ... \cup a_k$, then the partition $a_1 \cup a_2 \cup ... \cup a_k \cup \{r_1, ..., r_\ell, n+1\}$ is clearly a valid partition in X_{n+1} and uniquely defined. Therefore f is bijective and we are done. □

Problem 3. *For each $n \in \mathbb{N}$, prove that the number p_{odd} of partitions of n into odd parts equals the number $q(n)$ of partitions of n into distinct parts.*

Proof. We proceed via generating functions. Let a_n be the number of partitions of n into distinct parts and suppose $A(x) = \sum_{i=0}^{\infty} a_i x^i$ is the generating function for a_n. As an integer may appear at most once in a partition of n, $A(x)$ must be composed of only factors in the form $(1 + x^k)$ for all positive integers k. Therefore,

$$A(x) = \prod_{k=1}^{\infty}(1 + x^k) = \prod_{k=1}^{\infty} \frac{1 - x^{2k}}{1 - x^k} = \prod_{k \text{ odd}} \frac{1}{1 - x^k}.$$

The last equality is due to terms of the form $1 - x^{\ell}$ in the denominator for ℓ even cancelling with the term of the form $1 - x^{2 \cdot \frac{\ell}{2}}$ in the numerator. However, the last product may be represented as

$$\prod_{k \text{ odd}} (1 + x^k + x^{2k} + \ldots).$$

This representations tells us that we can make partitions out of any number of odd integers, where the term x^{ak} represents using k a total of a times in the partitions. This means that $A(x)$ is the generating function for $q(n)$ in addition to p_{odd}, implying they are equal. □

Problem 4. *Prove that, for every* $n \in \mathbb{N}$, *the following identity holds:*

$$\prod_{i=1}^{n}(1 + xq^i) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k+1}{2}} x^k.$$

Proof. We claim that both sides count the same event. Consider any term $c_{a,b}x^a q^b$ from the expansion of the left hand side. There must be exactly a terms of the form xq^i to produce the exponent x^a and further, the sum of the exponents of all the q^i terms must be b. Additionally, all q^i are distinct. Therefore, $c_{a,b}$ represents the number of partitions of b into a distinct parts, each of which is at most n. We wish to show the same is true of the right hand side.

Let $p(i, j, k)$ represent the number of partitions of k into at most j parts, each of which is at most i. From lecture, $\sum_{k \geq 0} p(i, j, k)q^k = \binom{i+j}{j}_q$. Plugging this into the right hand side gives

$$\sum_{k=0}^{n} x^k q^{\binom{k+1}{2}} \sum_{i \geq 0} p(n - k, k, i)q^i.$$

Fixing k and i, we obtain a term of the form $p(n - k, k, i)x^k q^{i+\binom{k+1}{2}}$ for $i \geq 0$. Consider some partition $p_1 + p_2 + \ldots + p_j = i$ with $p_1 \geq p_2 \geq \ldots \geq p_j$ such that $j \leq n$ and $p_1 \leq n - k$. If $j < k$, then let elements $p_{j+1}, p_{j+2}, \ldots, p_k$ all be equal to 0. Then, take $p'_i = p_i + i$, and so

$$\sum_{m=1}^{k} p'_m = \sum_{m=1}^{k} (p_m + m) = \binom{k+1}{2} + i,$$

meaning that the p'_m form a partition of $i + \binom{k+1}{2}$ into k distinct elements. Further, the largest that p'_1 can be is $n - k + k = n$, meaning that the p' form a unique partition that represents splitting $\binom{k+1}{2} + i$ into k distinct parts of at most n, for any i and k. This process is reversible, meaning that $c_{k,\binom{k+1}{2}+i}$ is the same as $p(n - k, k, i)$, and so the two sides represent the same function. □

Problem 5. *For* $n \in \mathbb{N}$, *what number of cycles do we expect when we take at random a permutation in* S_n*?*

Proof. Let X_k denote the random variable equal to the number of cycles of length k in a randomly chosen permutation in S_n. The sum $\sum_{k=1}^n X_k$ denotes the total number of cycles in the chosen permutation. Then, from linearity of expectation,

$$\mathbb{E}\left[\sum_{k=1}^n X_k\right] = \sum_{k=1}^n \mathbb{E}[X_k].$$

Now, we wish to count the number of cycles of length k across all $n!$ permutations. There are $\binom{n}{k}$ ways to pick k elements for a cycle, and $(k-1)!$ distinct ways to arrange the elements in a length k cycle. Therefore, there are a total of $\binom{n}{k}(k-1)!(n-k)! = \frac{n!}{k}$ cycles of length k across all permutations in S_n. Thus, $\mathbb{E}[X_k] = \frac{\frac{n!}{k}}{n!} = \frac{1}{k}$. This gives an answer of

$$\sum_{k=1}^n \mathbb{E}[X_k] = \sum_{k=1}^n \frac{1}{k} = H_n,$$

where H_n is the nth Harmonic number. □

Problem 6. *Let $I(n,j)$ be the number of permutations in S_n with no cycles of length greater than j. Prove the following recurrence identity:*

$$I(n+1,j) = \sum_{k=n-j+1}^{n} (n)_{n-k} I(k,j),$$

where $(n)_k := n(n-1)...(n-k+1)$.

Proof. Let $T_{n,j}$ be the set of permutations in S_n with no cycles of length greater than j. We claim that the number of permutations $\sigma \in T_{n+1,j}$ where $n+1$ is in a cycle of length $\ell \leq j$ is $(n)_{\ell-1} I(n-\ell+1,j)$. To get this, note that there are $\binom{n}{\ell-1}$ ways to pick the remaining $\ell-1$ elements in the same cycle as $n+1$, and $(\ell-1)!$ ways to arrange these elements within the cycle. Further, there are $I((n+1)-\ell,j)$ ways to permute the remaining $n+1-\ell$ elements into cycles of length at most j. Combining this, there are

$$\binom{n}{\ell-1}(\ell-1)! I(n+1-\ell,j) = n(n-1)(n-2)...(n-\ell+2) I(n+1-\ell,j) = (n)_{\ell-1} I(n+1-\ell,j)$$

total permutations with $n+1$ in a cycle of length ℓ. Summing this over all possible ℓ will give every potential permutation in $T_{n,j}$, thus we obtain

$$I(n+1,j) = \sum_{\ell=1}^{j} (n)_{\ell-1} I(n+1-\ell,j) = \sum_{k=n-j+1}^{n} (n)_{n-k} I(k,j),$$

where the last equality comes from a change of bounds. □

Problem 7. *For $n \in \mathbb{N}$ with $n \geq 2$, let $a(n,k)$ be the number of permutations in S_n with k cycles in which the entries 1 and 2 are in the same cycle. Prove the following identity:*

$$\sum_{k=1}^n a(n,k) x^k = x(x+2)...(x+n-1).$$

Proof. Let $c(n,k)$ denote the number of permutations in S_n that contain exactly k cycles. We claim that $c(n,k) = a(n,k) + a(n,k-1)$. In other words, it is sufficient to find a bijection between the number of permutations in S_n with exactly $k-1$ cycles that contain both 1 and 2 in a single cycle with those permutations in S_n that have k cycles with 1 and 2 in distinct cycles. Let X_k be the set of permutations with k cycles where 1 and 2 are in distinct cycles and Y_k be the set of permutations with k cycles where 1 and 2 are in the same cycle. Define $f : Y_{k-1} \to X_k$ as follows. Given a $y \in Y_{k-1}$, let the cycle 1 and 2 are in be $(1, a_1, ..., a_p, 2, b_1, ..., b_q)$. This cycle can have a unique split at a_p, the element just before 2 in the cycle and also after b_q to give us two distinct cycles $(1, a_1, ..., a_p)$ and $(2, b_1, ..., b_q)$. Further, this new permutation is an element of X_k, thus every element in Y_{k-1} has a unique mapping to Y_k under f.

For the inverse, it is not hard to see that we can take the two distinct cycles 1 and 2 would be in, namely $(1, a_1, ..., a_p)$ and $(2, b_1, ..., b_q)$, and merge them in the same way to get the cycle $(1, a_1, ..., a_p, 2, b_1, ..., b_q)$, thus meaning that f has an inverse map. As such, f is a bijective function and $c(n, k) = a(n, k)+a(n, k-1)$.

It was shown in lecture that

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)...(x+n-1).$$

Using this, we can break apart $c(n, k)$ to get the following:

$$\sum_{k=1}^{n} c(n,k)x^k = c(n,1)x + \sum_{k=2}^{n} c(n,k)x^k = a(n,1)x + \sum_{k=2}^{n}(a(n,k) + a(n,k-1))x^k =$$

$$\sum_{k=1}^{n} a(n,k)x^k + \sum_{k=2}^{n} a(n,k-1)x^k = \left(\sum_{k=1}^{n} a(n,k)x^k\right)(x+1).$$

With the fact from lecture, we find that

$$\sum_{k=1}^{n} a(n,k)x^k = x(x+2)(x+3)...(x+n-1).$$

□

Problem 8. *Each person in a group of n friends checks a hat and an umbrella when entering a restaurant. When they leave, each of them is given back at random a hat and an umbrella (from the same set of articles they had already checked upon entrance). In how many ways can none of the friends get back her/his own hat or umbrella?*

Proof. For $i \in [n]$, let A_i denote the set of events where person i gets both their hat and umbrella back. Using Sieve, we can count the total number of ways at least 1 person gets both their own hat and umbrella back. This gives

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\left|\bigcap_{i\in I,|I|=k,I\subseteq[n]} A_i\right|.$$

The $\binom{n}{k}$ coefficient comes from there being $\binom{n}{k}$ sets of size k for which we can take the intersection of for Sieve. Further, the intersection represents k people getting their articles back, and there are $(n-k)!$ ways to give back the remaining hats and $(n-k)!$ ways to give back the remaining umbrellas. Therefore, this quantity is

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}(n-k)!^2 = \sum_{k=1}^{n}(-1)^{k+1}\frac{n!(n-k)!}{k!}.$$

We want the complement of this expression, so we have to subtract this quantity from the total number of ways the hats and umbrellas can be given back, which is $(n!)^2$. Thus, the answer is

$$(n!)^2 - \sum_{k=1}^{n}(-1)^{k+1}\frac{n!(n-k)!}{k!} = \sum_{k=0}^{n}(-1)^{k}\frac{n!(n-k)!}{k!} = n!\sum_{k=1}^{n}(-1)^{k}\frac{(n-k)!}{k!}.$$ □